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# Painlevé analysis, auto-Bäcklund transformation and new analytic solutions for a generalized variable-coefficient Korteweg-de Vries (KdV) equation

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**Abstract.** There has been considerable interest in the study on the variable-coefficient nonlinear evolution equations in recent years, since they can describe the real situations in many fields of physical and engineering sciences. In this paper, a generalized variable-coefficient KdV (GvcKdV) equation with the external-force and perturbed/dissipative terms is investigated, which can describe the various real situations, including large-amplitude internal waves, blood vessels, Bose-Einstein condensates, rods and positons. The Painlevé analysis leads to the explicit constraint on the variable coefficients for such a equation to pass the Painlevé test. An auto-Bäcklund transformation is provided by use of the truncated Painlevé expansion and symbolic computation. Via the given auto-Bäcklund transformation, three families of analytic solutions are obtained, including the solitonic and periodic solutions.

**PACS.** 05.45.Yv Solitons – 02.30.Jr Partial differential equations – 52.35.Mw Nonlinear phenomena: waves, wave propagation, and other interactions (including parametric effects, mode coupling, ponderomotive effects, etc.) – 47.35.+i Hydrodynamic waves

## 1 Introduction

The Korteweg-de Vries (KdV) equation,

$$u_t + u\,u_x + u_{xxx} = 0\ , \tag{1}$$

is the prototype of the nonlinear evolution equations (NLEEs), where the wave amplitude $u(x,t)$ is a function of $x$ and $t$ (or, of the “space” and “time”, often scaled dimensionless). Although equation (1) arose originally from the long one-dimensional, small amplitude, surface gravity waves propagating in a shallow channel of water [1], it has been used to model a number of situations in physical science and engineering, including hydromagnetic waves, stratified internal waves, ion-acoustic waves, rotons, piasma physics, lattice dynamics and geophysical fluid dynamics [1–6] (and references therein).

It is well known that equation (1) is solvable by the inverse scattering method and then is completely integrable [1,7]. The Painlevé analysis, Bäcklund transformation, Lax pairs and similarity reductions have been obtained [8–10].

However, the physical situations in which the constant-coefficient NLEES arise tend to be much idealized, due to the assumption of their constant coefficients. When the media are inhomogeneous or the boundaries are nonuniform, as seen, e.g., in superconductors [11], plasmas [12] and optical-fiber communications [13], the variable-coefficient NLEEs are able to model various situations more realistically than their constant-coefficient counterparts [2,11–13]. An extension of KdV equation can be used to describe the large-amplitude internal waves in the coastal waters of the oceans, some of which have a distinct soliton character [6,14], where the coefficients of equation (1) vary with the vertical structure of the density and background flow, as observed in the Adriatic Sea [5], eastern Mediterranean [15], north west shelf of Australia [16], Baltic Sea [17], etc.

In this paper, we propose to study the Painlevé property and Bäcklund transformation of the generalized variable-coefficient KdV equation with the external-force and perturbed/dissipative terms [18],

$$u_t + f(t)\,u\,u_x + g(t)\,u_{xxx} + l(t)\,u = h(t)\ , \tag{2}$$

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which describes the solitonic structures in a varying-depth shallow-water tunnel, where $f(t)$, $g(t)$, $l(t)$ and $h(t)$ are real differentiable functions, with $f(t) \neq 0$ and $g(t) \neq 0$.

Many physical and mechanical situations governed by equation (2) have been seen, e.g., the propagation of pressure pulses in fluid-filled tubes of special value in arterial dynamics [19,20], the pulse wave propagation in blood vessels and dynamics in the circulatory system [21,22], matter waves and nonlinear atom optics enhanced by the observations of Bose-Einstein condensation in the weakly-interacting atomic gases [23–25], the nonlinear excitations of a Bose gas of impenetrable bosons with longitudinal confinement [26], the nonlinear waves in types of rods [27, 28], the infrared-absorption evidence of a positive-energy electronic bound state within the continuum above a potential well in the semiconductor heterostructures grown by molecular-beam epitaxy [29–31]. The details can be seen in reference [18] and references therein.

Special cases of equation (2), among others, include:

- The cylindrical case,

$$u_t + 6\,u\,u_x + u_{xxx} + \frac{u}{2\,t} = 0\,, \qquad (3)$$

  which is solvable by the inverse scattering method and then is completely integrable [32], with its Painlevé analysis, Lax representation, similarity reductions and analytic solutions obtained [33–35]. Recently equation (3) has bee used to model the cylindrical dust-acoustic and dust-ion-acoustic waves in space/laboratory dusty plasmas with varying velocity and amplitude [34].
- The forced case,

$$u_t + \alpha\,u\,u_x + \beta\,u_{xxx} = h(t)\,, \qquad (4)$$

  for $f(t) = \alpha$, $g(t) = \beta$ and $l(t) = 0$, with its exact solutions obtained by Jacobi elliptic function expansion [36].
- The perturbed case,

$$u_t + 6\,u\,u_x + u_{xxx} + l(t)\,u = 0, \qquad (5)$$

  for $f(t) = 6$, $g(t) = 1$ and $h(t) = 0$, which can pass the Painlevé test only when $l(t) \equiv 0$ or $l(t) = 1/[2\,(t-t_0)]$, while the corresponding equation is the standard KdV equation or cylindrical KdV equation respectively [37].
- The variable-coefficient case,

$$u_t + f(t)\,u\,u_x + g(t)\,u_{xxx} = 0, \qquad (6)$$

  for $h(t) = 0$ and $l(t) = 0$, which has the Painlevé property only when

$$g(t) = f(t)\,\left[a + b\int f(t)\,dt\right], \qquad (7)$$

  with its Bäcklund transformation, Lax pairs, similarity reductions and analytic solutions given [36,38,39].
- The special case,

$$u_t + a\,t^n\,u\,u_x + b\,t^m\,u_{xxx} = 0\,, \qquad (8)$$

  for $f(t) = a\,t^n$, $g(t) = b\,t^m$, $h(t) = 0$ and $l(t) = 0$, which posseses the Painlevé property whenever $m = n$ or $m = 2n+1$, with its Bäcklund transformation, Lax pairs and similarity reductions obtained [40].

This paper is arranged as follows. In Section 2, the Painlevé test is extended to equation (2) in order to obtain the constraints on the variable coefficients for equation (2) to posses the Painlevé property. In Section 3, an auto-Bäcklund transformation of equation (2) is provided via the Painlevé truncation and symbolic computation [2,11–13,41]. Furthermore, based on the given auto-Bäcklund transformation, three families of the new analytic solutions are obtained, including the solitonic and periodic solutions. Final section will be our discussions and conclusions.

## 2 Painlevé analysis and auto-Bäcklund transformation

A partial differential equation (PDE) is said to possess the Painlevé property if its solutions are single valued about the movable, singularity manifold and the singularity manifold is noncharacteristic [8]. Following the approach of the Painlevé PDE test [8] and the simplified Kruskal ansatz [42], we assume the solutions of equation (2) in a generalized Laurent expansion with the form,

$$u(x,t) = \phi^p(x,t)\sum_{j=0}^{\infty} u_j(t)\phi^j(x,t) \quad \text{with} \quad \phi(x,t) = x + \psi(t), \qquad (9)$$

where $\psi(t)$ is an arbitrary function of $t$, and $u_j(t)(j = 0,1,2,\cdots)$ are analytic functions of $t$, in the neighbourhood of a noncharacteristic movable singularity manifold defined by $\phi(x,t) = 0$. Substituting expression (9) into equation (2) and equating coefficients of like powers of $\phi$ determine $p$ and define recursion relations for $u_j(j = 0,1,2,\cdots)$. The Painlevé property requires that $p$ is a negative integer and the compatibility conditions are identically satisfied.

The leading-order analysis gives that $p = -2$ and

$$u_0 = -12\frac{g}{f}. \qquad (10)$$

The recursion relations are found to be

$$(j+1)(j-4)(j-6)u_j = F_j, \qquad (11)$$

where

$$F_j = \begin{cases} -\dfrac{1}{g}\Big[u_{j-3,t} + (j-4)u_{j-2}\psi_t \\ \quad + f\displaystyle\sum_{k=1}^{j-1}(k-2)u_k u_{j-k} + l u_{j-3}\Big]\,, & j \neq 5 \\ -\dfrac{1}{g}\Big[u_{j-3,t} + (j-4)u_{j-2}\psi_t \\ \quad + f\displaystyle\sum_{k=1}^{j-1}(k-2)u_k u_{j-k} + l u_{j-3}\Big] + \dfrac{h}{g}, & j = 5 \end{cases}$$

for $j \geq 0$ (define $u_j = 0$ for $j < 0$). We note that the resonances occur at $j = -1$, 4 and 6, while $j = -1$ corresponds to the arbitrariness of the function $\psi(t)$. Therefore there are two compatibility conditions at $j = 4$ and 6.

Putting $j = 1, 2, \cdots, 6$ in (11) and using (10), we get

$$j = 1: \qquad u_1 = 0; \tag{12}$$

$$j = 2: \qquad u_2 = -\frac{1}{f}\psi_t; \tag{13}$$

$$j = 3: \qquad u_3 = \frac{1}{g}\left[\left(\frac{g}{f}\right)_t + \frac{gl}{f}\right] = \left[\frac{1}{g}\left(\frac{g}{f}\right)_t + \frac{l}{f}\right]; \tag{14}$$

$$j = 4: \quad 0 \cdot u_4 = -\frac{1}{g}[u_{1,t} + 0 \cdot u_2\psi_t + f(-u_1u_3 + u_3u_1) + lu_1]; \tag{15}$$

$$j = 5: \qquad u_5 = \frac{1}{6g}(u_{2,t} + u_3\psi_t + fu_2u_3 + lu_2) - \frac{h}{g}; \tag{16}$$

$$j = 6: \quad 0 \cdot u_6 = -\frac{1}{g}[u_{3,t} + 2u_4\psi_t + f(-u_1u_5 + u_3^2 + 2u_2u_4 + 3u_5u_1) + lu_3]. \tag{17}$$

It is easily seen that the compatibility condition at $j = 4$ is satisfied identically, and the compatibility condition at $j = 6$ will give the constraint on the variable coefficients for equation (2) to have the Painlevé property.

Substituting (12)–(14) into (17) yields the compatibility condition at $j = 6$,

$$\left[\frac{1}{g}\left(\frac{g}{f}\right)_t + \frac{l}{f}\right]_t + f\left[\frac{1}{g}\left(\frac{g}{f}\right)_t + \frac{l}{f}\right]^2 + l\left[\frac{1}{g}\left(\frac{g}{f}\right)_t + \frac{l}{f}\right] = 0. \tag{18}$$

We make the transformation

$$v = \frac{1}{g}\left(\frac{g}{f}\right)_t + \frac{l}{f}, \tag{19}$$

which brings equation (18) into the form of Bernoulli equation [44],

$$v_t + f\,v^2 + l\,v = 0. \tag{20}$$

**Case 2-I.** $v = 0$:

In this case, equation (20) leads to

$$\left(\frac{g}{f}\right)_t + l\,\frac{g}{f} = 0\,, \tag{21}$$

and then we have

$$g(t) = c_1\,f(t)\,e^{-\int l(t)\,dt}, \tag{22}$$

with $c_1 \neq 0$ as a constant of integration.

**Case 2-II.** $v \neq 0$:

Solving equation (20) gives rise to

$$v = \frac{e^{-\int l\,dt}}{c_2 + \int f e^{-\int l\,dt}\,dt}, \tag{23}$$

where $c_2$ is a constant of integration. Substituting this back into equation (18), after some manipulations, we get

$$g = c_3\,f\,e^{-\int l\,dt}\left(c_2 + \int f\,e^{-\int l\,dt}\,dt\right), \tag{24}$$

with $c_3 \neq 0$ as a constant of integration.

Combining the above two cases, we thus obtain the explicit constraint on the variable coefficients $f(t)$, $g(t)$ and $l(t)$ for equation (2) to pass the Painlevé test

$$g(t) = f(t)\,e^{-\int l(t)\,dt}\left(a + b\int f(t)\,e^{-\int l(t)\,dt}\,dt\right), \quad a^2 + b^2 \neq 0\,. \tag{25}$$

Furthermore, it is easily found that Constraint (25) is the same as that of the transformations from the equation (2) to the cylindrical KdV and KdV equations which are completely integrable [18]. Therefore, equation (2) is predicted to be completely integrable if and only if the variable coefficients of equation (2) satisfy Constraint (25).

## 3 Auto-Bäcklund transformation and solutions

For the derivation of auto-Bäcklund transformation of equation (2), we must work with the general form $\phi(x,t) = 0$ of the noncharacteristic singularity manifold instead of the simplified Kruskal ansatz $\phi(x,t) = x + \psi(t)$ in the above Painlevé analysis [43]. With leading-order analysis, we obtain the truncated Painlevé expansion at the constant level term as

$$u(x,t) = \phi^{-2}(x,t)\sum_{j=0}^{2} u_j(x,t)\phi^j(x,t) = u_0(x,t)\phi^{-2}(x,t) + u_1(x,t)\phi^{-1}(x,t) + u_2(x,t)\,. \tag{26}$$

Substituting (26) into equation (2) and making the coefficients of like powers of $\phi$ vanish yield

$$\phi^{-5}: -2\,f\,u_0^2\,\phi_x - 24\,g\,u_0^2\,\phi_x^3 = 0 \quad \Rightarrow \quad u_0 = -12\,\frac{g}{f}\,\phi_x^2\,, \tag{27}$$

$$\phi^{-4}: f\,(u_0u_{0,x} - 3\,u_0u_{0,x}\phi_x) + 6\,g\,(3\,u_{0,x}\,\phi_x^2 + 3\,u_0u_{0,x}\,\phi_x\,\phi_{x,x} - u_1\,\phi_x^3) = 0 \quad \Rightarrow \quad u_1 = 12\,\frac{g}{f}\,\phi_{xx}\,, \tag{28}$$

$$\phi^{-3}: -2u_0\phi_t + f(u_0u_{1,x} + u_{0,x}u_1 - u_1^2\phi_x - 2u_0u_2\phi_x) - 2\,g\,(3\,u_{0,xx}\,\phi_x + 3\,u_{0,x}\,\phi_{xx} + u_0\,\phi_{xxx} - 3\,u_{1,x}\,\phi_x^2 - 3\,u_1\,\phi_x\,\phi_{xx}) = 0\,, \tag{29}$$

$$\phi^{-2}: (u_{0,t} - u_1\,\phi_t) + f\,(u_0 u_{2,x} + u_1 u_{1,x} + u_{0,x} u_2 - u_1 u_2\,\phi_x) + g\,(u_{0,xxx} - 3\,u_{1,xx}\,\phi_x - 3\,u_{1,x}\,\phi_{xx} - u_1\,\phi_{xxx}) + l\,u_0 = 0\,, \tag{30}$$

$$\phi^{-1}: u_{1,t} + f\,(u_1 u_{2,x} + u_{1,x} u_2) + g\,u_{1,xxx} + l\,u_1 = 0\,, \tag{31}$$

$$\phi^{0}: u_{2,t} + f\,u_2 u_{2,x} + g\,u_{2,xxx} + l\,u_2 = h\,. \tag{32}$$

By making use of equations (27) and (28), equations (29)–(31) become

$$2\,\phi_x(\phi_x\,\phi_t + f\,u_2\,\phi_x^2 + 4\,g\,\phi_x\,\phi_{xxx} - 3\,g\,\phi_{xx}^2) = 0\,, \tag{33}$$

$$\phi_x\left\{\left[\frac{f}{g}\left(\frac{g}{f}\right)_t + l\right]\phi_x + \phi_{xt} + f\,u_2\,\phi_{xx} + g\,\phi_{xxxx}\right\} + \frac{\partial}{\partial x}\Big(\phi_x\,\phi_t + f\,u_2\,\phi_x^2 + 4\,g\,\phi_x\,\phi_{xxx} - 3\,g\,\phi_{xx}^2\Big) = 0, \tag{34}$$

$$\frac{\partial}{\partial x}\left\{\left[\frac{f}{g}\left(\frac{g}{f}\right)_t + l\right]\phi_x + \phi_{xt} + f\,u_2\,\phi_{xx} + g\,\phi_{xxxx}\right\} = 0 \tag{35}$$

respectively. It is easily seen that equations (29)–(32) are satisfied, provided that ($\phi_x \neq 0$)

$$\phi_x\,\phi_t + f\,u_2\,\phi_x^2 + 4\,g\,\phi_x\,\phi_{xxx} - 3\,g\,\phi_{xx}^2 = 0\,, \tag{36}$$

$$\left[\frac{f}{g}\left(\frac{g}{f}\right)_t + l\right]\phi_x + \phi_{xt} + f\,u_2\,\phi_{xx} + g\,\phi_{xxxx} = 0\,, \tag{37}$$

$$u_{2,t} + f\,u_2 u_{2,x} + g\,u_{2,xxx} + l\,u_2 = h\,. \tag{38}$$

Therefore, we obtain an auto-Bäcklund transformation of equation (2) as follows,

$$u = 12\,\frac{g}{f}\frac{\partial^2}{\partial x^2}\ln\phi + u_2, \tag{39}$$

where $\phi$ satisfies equations (36) and (37), and $u_2$ is a solution of equation (2).

Via the above auto-Bäcklund transformation and choosing the different $u_2(x,t)$ and $\phi(x,t)$, one can obtain various solutions. In illustration, next we will provide three families of the analytic solutions, including the solitonic and periodic solutions.

**Case 3-I.**

Setting

$$\phi(x,t) = 1 + e^{p(t)\,x + q(t)}\,, \tag{40}$$

$$u_2(x,t) = r(t)\,x + s(t)\,, \tag{41}$$

where the differentiable functions $p(t) \neq 0$, $q(t)$, $r(t)$ and $s(t)$ are to be determined, and substituting (40) and (41) into equations (36)–(38) give rise to a set of the three equations:

$$(p' + f\,p\,r)\,x + (q' + f\,p\,s + g\,p^3) = 0\,, \tag{42}$$

$$p\left[(p' + f\,p\,r)\,x + (q' + f\,p\,s + g\,p^3)\right] + p' + p\left[\frac{f}{g}\left(\frac{g}{f}\right)_t + l\right] = 0\,, \tag{43}$$

$$(r' + f\,r^2 + l\,r)\,x + (s' + f\,r\,s + l\,s - h) = 0\,. \tag{44}$$

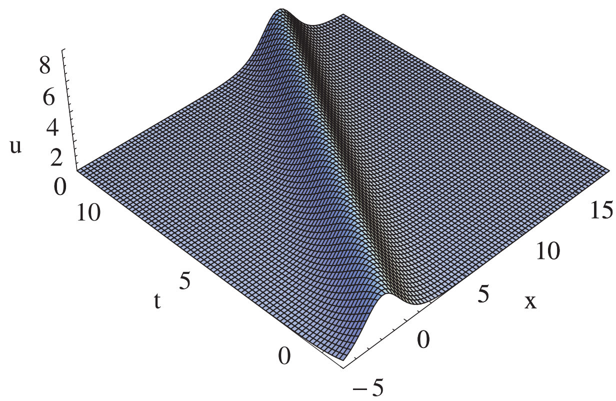


**Fig. 1.** The traveling wave solution surface $u(x,t)$ via Expression (49) with $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 = 0$, $\alpha_4 = 1$, $f(t) = 1$, $l(t) = 0$ and $h(t) = 0$.

Solving equations (42)–(44) will give two family solutions:

When $r(t) = 0$, we have

$$s(t) = e^{-\int l(t)\,dt}\Big(\alpha_1 + \int h(t)\,e^{\int l(t)\,dt}\,dt\Big)\,, \tag{45}$$

$$p(t) = \alpha_2\,, \tag{46}$$

$$q(t) = -\alpha_2\int f(t)\,e^{-\int l(t)dt}\left[\alpha_1 + \alpha_2^2\,\alpha_4 + \int h(t)\,e^{\int l(t)\,dt}\,dt\right]dt + \alpha_2\,\alpha_3 \tag{47}$$

with the consistency condition is

$$g(t) = \alpha_4\,f(t)\,e^{-\int l(t)\,dt}\,, \tag{48}$$

where $\alpha_1$, $\alpha_2 \neq 0$, $\alpha_3$ and $\alpha_4 \neq 0$ are all constants of integration. We note that Constrain (48) is the special case of Constrain (25) obtained by Painlevé test with $b = 0$.

We then obtain the first family of the exact analytic solutions of equation (2),

$$u^I(x,t) = e^{-\int l(t)\,dt}\Big(\alpha_1 + \int h(t)\,e^{\int l(t)\,dt}\,dt\Big) + 3\,\alpha_2^2\,\alpha_4\,e^{-\int l(t)\,dt} \times \mathrm{sech}^2\left\{\frac{\alpha_2}{2}\left[x - \int f(t)\,e^{-\int l(t)dt} \times\Big(\alpha_1 + \alpha_2^2\,\alpha_4 + \int h(t)\,e^{\int l(t)\,dt}\,dt\Big) + \alpha_3\right]\right\}, \tag{49}$$

which are the solitonic solutions.

$$r(t) = \frac{e^{-\int l(t)\,dt}}{\alpha_5 + \int f(t)\, e^{-\int l(t)\,dt}\,dt}\,, \tag{50}$$

$$s(t) = \frac{e^{-\int l(t)\,dt}\Big[\alpha_6 + \int h(t)\, e^{\int l(t)\,dt}(\alpha_5 + \int f(t)\, e^{-\int l(t)\,dt}\,dt)\,dt\Big]}{\alpha_5 + \int f(t)\, e^{-\int l(t)\,dt}\,dt}\,, \tag{51}$$

$$p(t) = \frac{\alpha_7}{\alpha_5 + \int f(t)\, e^{-\int l(t)\,dt}\,dt}\,, \tag{52}$$

$$q(t) = \alpha_7\,\alpha_9 - \alpha_7 \int h(t)\, e^{\int l(t)\,dt}\,dt + \frac{\alpha_7\Big(\alpha_6 + \alpha_7^2\,\alpha_8 + \int h(t)\, e^{\int l(t)\,dt}(\alpha_5 + \int f(t)\, e^{-\int l(t)\,dt}\,dt)\,dt\Big)}{\alpha_5 + \int f(t)\, e^{-\int l(t)\,dt}\,dt}\,, \tag{53}$$

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$$u^{II}(x,t) = \frac{e^{-\int l(t)\,dt}\Big[x + \alpha_6 + \int h(t)\, e^{\int l(t)\,dt}(\alpha_5 + \int f(t)\, e^{-\int l(t)\,dt}dt)\,dt\Big]}{\alpha_5 + \int f(t)\, e^{-\int l(t)\,dt}\,dt} + \frac{3\alpha_7^2\,\alpha_8\, e^{-\int l(t)\,dt}}{\alpha_5 + \int f(t)\, e^{-\int l(t)\,dt}\,dt}\,\mathrm{sech}^2\Bigg\{\frac{\alpha_9 - \alpha_7\int h(t)\, e^{\int l(t)\,dt}\,dt}{2} + \frac{\alpha_7\Big[x + \alpha_6 + \alpha_7^2\,\alpha_8 + \int h(t)\, e^{\int l(t)\,dt}(\alpha_5 + \int f(t)\, e^{-\int l(t)\,dt}dt)\,dt\Big]}{2\,(\alpha_5 + \int f(t)\, e^{-\int l(t)\,dt}\,dt)}\Bigg\}\,, \tag{55}$$

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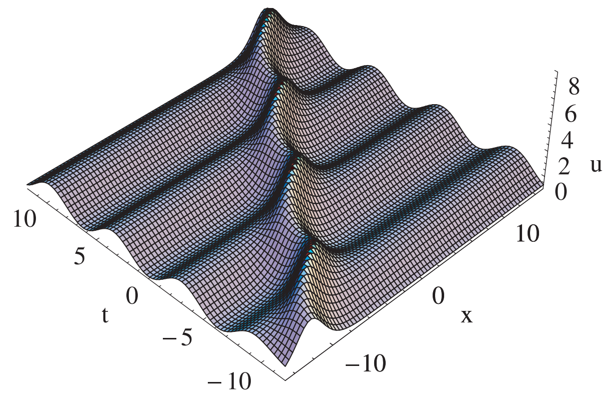


**Fig. 2.** The solution surface $u(x,t)$ via Expression (49) with the same choices as Figure 1 except that $h(t) = 0.8\,\sin t$.

When $r(t) \neq 0$, we have

*see equations (50–53) above*

with the consistency condition is

$$g(t) = \alpha_8\, f(t)\, e^{-\int l(t)\,dt}\Big(\alpha_5 + \int f(t)\, e^{-\int l(t)\,dt}\,dt\Big)\,, \tag{54}$$

where $\alpha_5,\ \alpha_6,\ \alpha_7 \neq 0,\ \alpha_8 \neq 0$ and $\alpha_9$ are all constants of integration. We note that Constrain (54) is also the special case of Constrain (25) obtained by Painlevé test but with $b \neq 0$.

Therefore, we obtain the second family of the analytic solutions of equation (2),

*see equation (55) above*

which are the solitonic solutions.

**Case 3-II.**

If setting

$$\phi(x,t) = 1 + e^{\mathrm{i}[m(t)\,x + n(t)]}\,, \tag{56}$$

$$u_2(x,t) = u_2(t)\,, \tag{57}$$

where the real differentiable functions $m(t) \neq 0,\ n(t)$ and $u_2(t)$ are to be determined, then $m(t),\ n(t)$ and $u_2(t)$ satisfy the following equations:

$$m'\,x + n' + f\,u_2\,m - g\,m^3 = 0\,, \tag{58}$$

$$\mathrm{i}\left[\frac{f}{g}\left(\frac{g}{f}\right)_t + l\right] - (m'\,x + n' + f\,u_2\,m - g\,m^3) = 0\,, \tag{59}$$

$$u_2' + l\,u_2 = h\,. \tag{60}$$

After some calculations, we obtain the third family of the analytic solution of equation (2),

$$u^{III}(x,t) = e^{-\int l(t)\,dt}\Big(\alpha_9 + \int h(t)\, e^{\int l(t)\,dt}\,dt\Big) - 3\,\alpha_4\,\alpha_{10}^2\, e^{-\int l(t)\,dt} \times \mathrm{sec}^2\Bigg\{\frac{\alpha_{10}}{2}\Big[x + \int f(t)\, e^{-\int l(t)dt}\Big(\alpha_4\,\alpha_{10}^2 - \alpha_9 - \int h(t)\, e^{\int l(t)\,dt}\,dt\Big)\,dt\Big] + \alpha_{11}\Bigg\}\,, \tag{61}$$

which are the periodic solutions, where the variable coefficients $f(t),\ g(t),\ l(t)$ satisfy Constraint (48), and $\alpha_9,\ \alpha_{10} \neq 0,\ \alpha_{11}$ are all arbitrary constants.

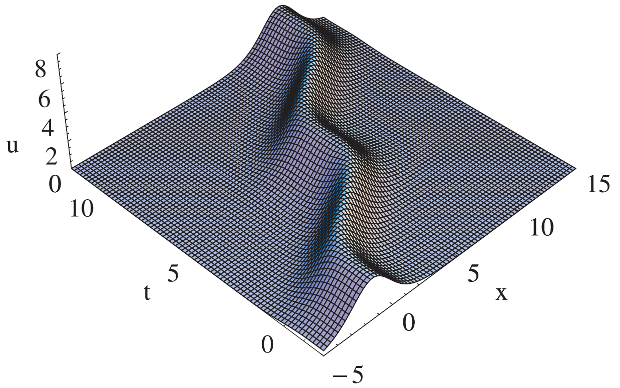


**Fig. 3.** The solution surface $u(x, t)$ via Expression (49) with the same choices as Figure 1 except that $f(t) = 1 + \sin t$.

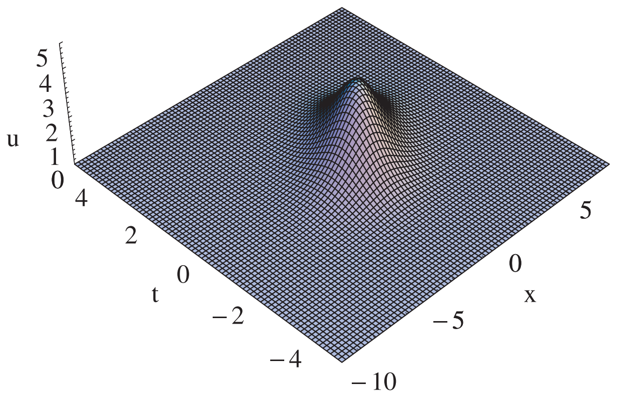


**Fig. 4.** The solution surface $u(x, t)$ via Expression (49) with the same choices as Figure 1 except that $l(t) = 1$.

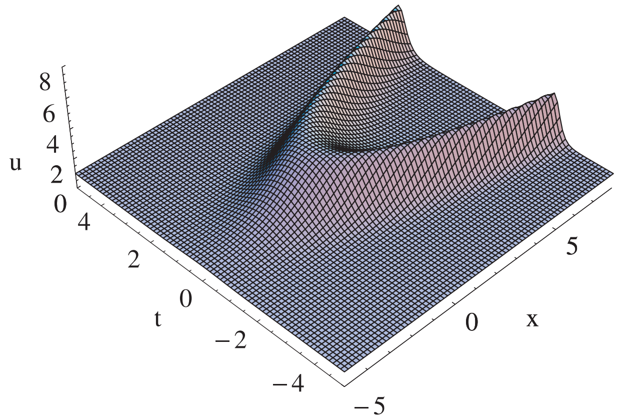


**Fig. 5.** The solution surface $u(x, t)$ via Expression (49) with the same choices as Figure 1 except that $\alpha_1 = 1$ and $f(t) = t$.

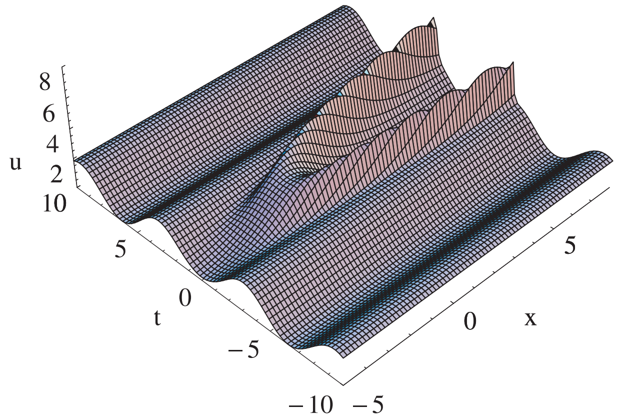


**Fig. 6.** The solution surface $u(x, t)$ via Expression (49) with the same choices as Figure 1 except that $\alpha_1 = 1,\ f(t) = t$ and $h(t) = \sin t$.

## 4 Discussions and conclusions

The variable-coefficient nonlinear evolution equations, although their coefficient functions often make the studies hard, are of current interests since they are able to describe the real situations in many fields of physical and engineering sciences. In this paper, we have studied the generalized variable-coefficient KdV equation, i.e., equation (2), with the external-force and perturbed/dissipative terms, which can model the various real situations, including large-amplitude internal waves, blood vessels, Bose-Einstein condensates, rods and positons. The Painlevé analysis leads to the explicit constraint on the variable coefficients for such a equation to pass the Painlevé test. An auto-Bäcklund transformation is presented by use of the truncated Painlevé expansion and symbolic computation. Via the given auto-Bäcklund transformation, three families of the new analytic solutions are obtained , including the solitonic and periodic solutions. Now let us conclude and discuss our results as below:

1. We have applied directly the PDE Painlevé test to equation (2) and provided Constrain (25) which is the necessary and sufficient condition for equation (2) to have the Painlevé property. We have noted that Constraint (25) coincides with that of the transformations from the equation (2) to the cylindrical KdV and KdV equations which both are completely integrable. Thus, equation (2) is predicted to be completely integrable if and only if the variable coefficients of equation (2) satisfy Constraint (25).
2. We have also noted that Constraint (25) depends only on the coefficient functions $f(t)$, $g(t)$ and $l(t)$, and has nothing to do with the external-force term $h(t)$ of the system. The reason is that $h(t)$, existing in the space-time $(x, t)$ and obviously affecting the field $u(x, t)$, can be "absorbed" by a proper transformation/scaling from $u(x, t)$ to $U(X, T)$, so that we can discuss, in the new space-time $U(X, T)$, the effect of the perturbation on the new field $U$, without the external-force term any more. In illustration, equation (2) can be transformed into the equation

$$U_t + f(t)\, U\, U_X + g(t)\, U_{xxx} + l(t)U = 0, \qquad (62)$$

with the transformation [18]

$$u(x, t) = U(X, t) + B(t)\ , \qquad (63)$$

where

$$B(t) = e^{-\int l(t)\,dt} \int e^{\int l(t)\,dt} h(t)\,dt\,,$$
$$X = x - \int f(t) B(t)\,dt.$$

Hence, equation (2) is equivalent to equation (62) in the sense of [45].

3. The more general KdV equation with the time and space depending coefficients

$$u_t + a(t)\,u + (b(t,x)\,u)_x + c(t)\,u\,u_x + d(t)\,u_{xxx} + e(x,t) = 0, \quad (64)$$

has been considered, with its auto-Bäcklund transformation, Lax pairs given in references [46,47]. Using the property of Lax pairs, reference [46] has obtained the condition

$$b_t + (a - Lc)\,b + b\,b_x + d\,b_{xxx} = 2\,a\,h + h\,L\frac{d}{c^2} + \frac{dh}{dt} + c\,e + x\left(2\,a^2 + a\,L\frac{d^3}{c^4} + \frac{da}{dt} + L\frac{d}{c}\,L\frac{d}{c^2} + \frac{d}{dt}L\frac{d}{c}\right), \quad (65)$$

where $L = (d/dt)\ln$, and $h(t)$ is an arbitrary and sufficiently smooth function of $t$, which admits that equation (64) possesses the Painlevé property. We hereby note that Constraint (65) is implicit and not obtained directly by the Painlevé test, and that Constraint (65) is only sufficient condition for equation (64) to have the Painlevé property [46]. However, Constraint (25) obtained directly by the Painlevé test in this paper is given in explicit form, and is the necessary and sufficient condition for equation (2) to have the Painlevé property.

4. We have provided the auto-Bäcklund transformation (39). Via this transformation and choosing the different $u_2(x,t)$ and $\phi(x,t)$, one can obtain various solutions. In illustration, three families of analytic solutions have been presented, including the solitonic and periodic solutions, which are Solutions (49), (55) and (61). To our knowledge, these solutions have not been obtained before, and are expected to be of value for explaining the above various phenomena listed in the first section.
For Solutions (49) and (61), $e^{-\int l(t)\,dt}$ is the attenuation factor affecting the amplitude of the wave. The field $u(x,t)$ is exponentially sensitive to $-\int l(t)\,dt$, which represents the accumulated effect of perturbation over a time period. If $\int l(t)\,dt \gg 1$, the amplitude will decrease exponentially.
However, for Solution (55), the attenuation factor is $\frac{e^{-\int l(t)\,dt}}{\alpha_5 + \int f(t)\,e^{-\int l(t)\,dt}\,dt}$, which affects the amplitude of the wave.

5. To investigate the effects the coefficient functions have on Solution (49), we plot six graphs of the solution surfaces with some special choices for the coefficient functions and parameters listed in their captions. From those figures, one can easily discover that the variable coefficients make the solutions go beyond the traveling waves, and obviously change the shapes or sizes of the waves.

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